Mathematics: analysis and approaches

Standard level

Paper 1

Name

worked solutions

Date: ____________________

1 hour 30 minutes

SOLUTION KEY

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written in the answer boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches SL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

exam: 7 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

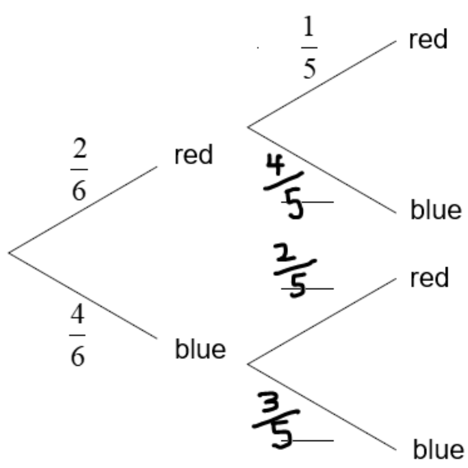
Section A (37 marks)

1. [Maximum mark: 6]

A bag contains 2 red balls and 4 blue balls. Two balls are selected at random without replacement.

(a) Complete the following diagram. [3]

Answers:



(b) Find the probability that exactly one of the selected balls is red. [3]

Solution:

Probability 1st ball is red (R) and 2nd ball is blue (B) $= \mathrm{P}(\mathrm{RB}) = \frac{2}{6} \cdot \frac{4}{5} = \frac{8}{30} = \frac{4}{15}$

Probability 1st ball is blue (B) and 2nd ball is red (R) $= \mathrm{P}(\mathrm{BR}) = \frac{4}{6} \cdot \frac{2}{5} = \frac{8}{30} = \frac{4}{15}$

Therefore, probability exactly one of the balls is red $= \mathrm{P}(\mathrm{RB}) + \mathrm{P}(\mathrm{BR}) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$

2. [Maximum mark: 4]

The equation $ax^2 + 3x + 2 = 0$, where a is a constant, has exactly one solution. Find the value of a.

Solution:

A quadratic equation, $ax^2 + bx + c = 0$, has exactly one solution when the discriminant, $b^2 - 4ac$, is zero.

$$b^2 - 4ac = 3^2 - 4 \cdot a \cdot 2 = 9 - 8a = 0 \quad \Rightarrow \quad a = \frac{9}{8}$$

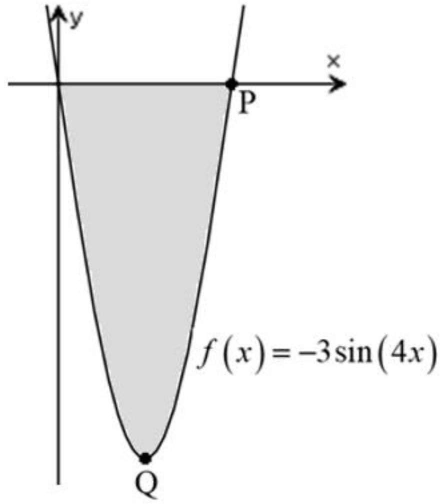
3. [Maximum mark: 6]

A portion of the graph of $f(x) = -3\sin(4x)$ is shown.
The point P is an x-intercept with coordinates $(p, 0)$.

(a) Find the value of p. [2]

(b) The point Q is a minimum. Write down the coordinates of Q. [2]

(c) Write down a definite integral, but do not evaluate it, that represents the shaded region bounded by f and the x-axis. [2]



Solution:

(a) $-3\sin(4p) = 0 \quad \Rightarrow \quad 4p = k\pi, k \in \mathbb{Z} \quad \Rightarrow \quad p = \frac{\pi}{4}$

(b) due to symmetry of the graph of f, the x-coordinate of Q must be one-half the x-coordinate of P;
and since the maximum value of $\sin(4x)$ is 1, then the minimum value of $-3\sin(4x)$ is -3
therefore, the coordinates of Q are $\left(\frac{\pi}{8}, -3\right)$

(c) area of shaded region $= \left| \int_0^{\frac{\pi}{4}} (-3\sin(4x))\,dx \right|$ note: absolute value required

4. [Maximum mark: 6]

The sum of the first three terms of an arithmetic sequence is 6 and the fourth term is 16.
Find the first term, u_1, and the common difference, d, of the sequence.

Solution:

$u_1 + (u_1 + d) + (u_1 + 2d) = 6 \quad \Rightarrow \quad 3u_1 + 3d = 6$ and $u_1 + 3d = 16 \quad \Rightarrow \quad u_1 = 16 - 3d$

Substituting gives $3(16 - 3d) + 3d = 6 \quad \Rightarrow \quad 48 - 6d = 6 \quad \Rightarrow \quad 6d = 42 \quad \Rightarrow \quad d = 7$

$u_1 = 16 - 3(7) = -5$ answers: $u_1 = -5$ and $d = 7$

5. [Maximum mark: 7]

(a) Given $f(x)=x^2+4x-10,\ x\le -2$ show that $f^{-1}(x)=-2-\sqrt{x+14},\ x\ge -14$. [4]

(b) The graphs of f and f^{-1} intersect at point C. Find the coordinates of C. [3]

Solution:

(a) $f(x)=x^2+4x-10,\ x\le -2 \Rightarrow y=x^2+4x-10$ switch domain & range: $x=y^2+4y-10$

solve for y using 'completing the square': $x=y^2+4y-10 \Rightarrow x=y^2+4y+4-14 \Rightarrow x+14=(y+2)^2$

$y+2=\pm\sqrt{x+14} \Rightarrow y=-2-\sqrt{x+14}$ (negative square root because range of f^{-1} is $y\le -2$)

thus, $f^{-1}(x)=-2-\sqrt{x+14},\ x\ge -14$ ***Q.E.D.***

(b) point C must be on the line $y=x$; hence, solve $x^2+4x-10=x$

$$x^2+3x-10=0 \Rightarrow x=\frac{-3\pm\sqrt{3^2-4(1)(-10)}}{2}=\frac{-3\pm\sqrt{49}}{2}=\frac{-3\pm 7}{2} \Rightarrow x=2 \text{ or } x=-5$$

$x\ne 2$ because the domain of f is $x\le -2$; hence, $x=-5$ and since C is on $y=x$ then $y=-5$

thus, the coordinates of C are $(-5,-5)$

6. [Maximum mark: 6]

Show that $\log_2\sqrt{8}+\log_b\sqrt{ab}=\dfrac{\ln(ab^4)}{\ln(b^2)}$

Solution: $\log_2\sqrt{8}+\log_b\sqrt{ab}=\log_2\left[\left(2^3\right)^{\frac{1}{2}}\right]+\log_b\left(\sqrt{a}\sqrt{b}\right)$

$$=\log_2\left(2^{\frac{3}{2}}\right)+\log_b\left(a^{\frac{1}{2}}\right)+\log_b\left(b^{\frac{1}{2}}\right)$$

$$=\frac{3}{2}+\frac{1}{2}\log_b a+\frac{1}{2}$$

$$=2+\frac{1}{2}\log_b a$$

$$=\frac{4\ln b}{2\ln b}+\frac{1}{2}\cdot\frac{\ln a}{\ln b}$$

$$=\frac{4\ln b+\ln a}{2\ln b}$$

$$=\frac{\ln(b^4)+\ln a}{\ln(b^2)}$$

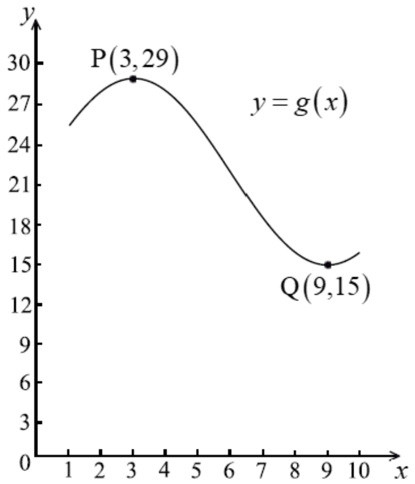
$$=\frac{\ln(ab^4)}{\ln(b^2)}$$ ***Q.E.D.***

Section B (43 marks)

7. [Maximum mark: 12]

Let $g(x)=a\cos[b(x+c)]+d,\ 1\le x\le 10$. The graph of $y=g(x)$ is shown below.

There is a maximum value of 29 at P when $x=3$, and a minimum value of 15 at Q when $x=9$.



(a) (i) Given $a>0$, find the value of a.

(ii) Show that $b=\frac{\pi}{6}$.

(iii) Find the value of d.

(iv) Write down the value of c. [7]

The graph of g undergoes three transformations. It is stretched horizontally by a scale factor of $\frac{1}{2}$, followed by a vertical translation of $+8$ units, followed by a horizontal translation of $+6$ units. The new transformed graph is the graph of the function h.

(b) Find the coordinates for the maximum point on the graph of h. [2]

(c) $h(x)$ can be expressed in the form $h(x)=g[B(x+C)]+D$. Find the value of B, the value of C, and the value of D. [3]

Solution:

(a) (i) $\frac{29-15}{2}=7$; hence, the amplitude of g is 7 and since the amplitude of $y=\cos x$ is 1 then $y=\cos x$ was vertically stretched by a factor of 7 to create graph of g; therefore, $a=7$

(ii) period of g is 12 since horizontal distance of 6 from P to Q is one-half the period

hence, $b=\frac{2\pi}{12}=\frac{\pi}{6}$ ***Q.E.D.***

(iii) $d=\frac{29+15}{2} \Rightarrow d=22$

(iv) $y=\cos x$ is translated 3 units to the right to create graph of g; therefore $c=-3$

(b) horizontal stretch with scale factor of $\frac{1}{2}$ moves x-coordinate of P to $\frac{3}{2}$

vertical translation of $+8$ moves y-coordinate of P to 37

horizontal translation of $+6$ moves x-coordinate of P to $\frac{15}{2}$; thus, maximum point is at $\left(\frac{15}{2},37\right)$

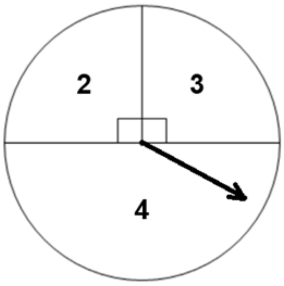
(c) horizontal stretch with scale factor $=\frac{1}{2} \Rightarrow B=2$

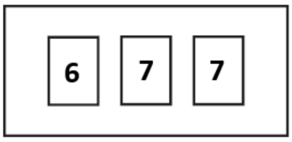
vertical translation of $+8 \Rightarrow D=8$

horizontal translation of $+6 \Rightarrow C=-6$ $h(x)=g[2(x-6)]+8$

8. [Maximum mark: 15]

A spinner consists of an arrow that rotates about the centre of a circle so that one of three numbers is randomly selected (see diagram below). There is also a box containing three numbered cards as shown below. S is the sum of two numbers – one selected randomly with the spinner and the other from randomly selecting one of the cards from the box.





(a) Write down the four different possible values of S. [2]

(b) Find the probability of each value of S. [5]

(c) Show that the expected value of S is $\frac{119}{12}$. [2]

(d) Anna plays a game where she wins \$15 if S is an even number and loses \$10 if S is an odd number. Sophie plays the game 12 times. Find the amount of money she expects to have at the end of the 12 games. [6]

Solution:

(a) possible values of S: $\underline{8, 9, 10, 11}$

$8 = 2 + 6$
$9 = 3 + 6$ OR $2 + 7$
$10 = 3 + 7$ OR $4 + 6$
$11 = 4 + 7$

(b) $P(S=8) = \frac{1}{4} \cdot \frac{1}{3} = \underline{\frac{1}{12}}$

$P(S=9) = \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{3}{12} = \underline{\frac{1}{4}}$

$P(S=10) = \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{4}{12} = \underline{\frac{1}{3}}$

$P(S=11) = \frac{1}{2} \cdot \frac{2}{3} = \frac{4}{12} = \underline{\frac{1}{3}}$

(c) $E(S) = \frac{1}{12}(8) + \frac{3}{12}(9) + \frac{4}{12}(10) + \frac{4}{12}(11)$

$= \frac{8}{12} + \frac{27}{12} + \frac{40}{12} + \frac{44}{12} \Rightarrow \underline{E(S) = \frac{119}{12}}$

(d) probability of even number (8 or 10) $= \frac{1}{12} + \frac{4}{12} = \frac{5}{12}$

probability of odd number (9 or 11) $= \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

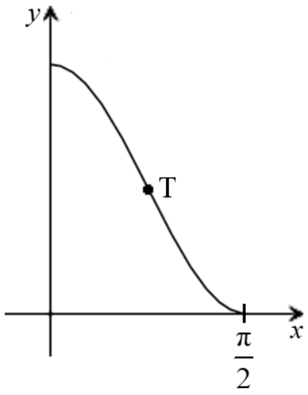
expected gain/loss from 1 game $= \frac{5}{12}(15) - \frac{7}{12}(10)$

expected amount of \$ at end of 12 games $= 12\left[\frac{5}{12}(15) - \frac{7}{12}(10)\right]$

$= 5(15) - 7(10) = 75 - 70 = \underline{\$5}$

9. [Maximum mark: 16]

A graph of the function $f(x) = 2\cos^2 x,\ 0 \le x \le \frac{\pi}{2}$ is shown below.



(a) Point T is a point of inflexion. Show that the coordinates of T are $\left(\frac{\pi}{4}, 1\right)$. [5]

(b) Line L is tangent to the graph of f at T. Find the equation of L and express it in the form $y = mx + c$. [4]

(c) Find the area of the region bounded by the x-axis, the y-axis and the graph of f. [7]

Solution:

(a) $f'(x) = 2(2\cos x)(-\sin x) = -4\sin x\cos x$

$f''(x) = -4(\cos x\cos x + \sin x(-\sin x)) = -4(\cos^2 x - \sin^2 x) = 0$; x is between 0 and $\frac{\pi}{2}$

$f''(x) = 0$ when $\cos^2 x = \sin^2 x \Rightarrow \cos x = \sin x \Rightarrow x = \frac{\pi}{4}$; $f\left(\frac{\pi}{4}\right) = 2\left[\cos\left(\frac{\pi}{4}\right)\right]^2 = 2\left[\frac{\sqrt{2}}{2}\right]^2 = 1$

Thus, the coordinates of the inflexion point T are $\left(\frac{\pi}{4}, 1\right)$ ***Q.E.D.***

(b) $f'\left(\frac{\pi}{4}\right) = -4\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = -4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -2$

equation of line L: $y - 1 = -2\left(x - \frac{\pi}{4}\right) \Rightarrow y = -2x + \frac{\pi}{2} + 1 \Rightarrow y = -2x + \frac{\pi + 2}{2}$

(c) area $= \int_0^{\frac{\pi}{2}} (2\cos^2 x)\,dx$

apply trig identity $\cos 2\theta = 2\cos^2\theta - 1$; hence, $2\cos^2\theta = 1 + \cos 2\theta$; substituting gives

area $= \int_0^{\frac{\pi}{2}} [1 + \cos(2x)]\,dx = x + \frac{1}{2}\sin(2x)\Big]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(0 + \frac{1}{2}\sin(0)\right) = \left(\frac{\pi}{2} + \frac{1}{2}\cdot 0\right) - \left(0 + \frac{1}{2}\cdot 0\right)$

Therefore, the area of the bounded region is $\frac{\pi}{2}$ square units